Generalized λ-Closed Sets and $(\lambda,\gamma)^*$-Continuous Functions

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Abstract— In this paper we introduce the concept of λ-open set and by using this set we define generalized λ-closed set we obtain some of its properties and also we define $(\lambda,\gamma)^*$-continuous function and study some of its basic properties.

Index Terms— s-operation, λ-open, generalized λ-closed, λ-$T_{1/2}$ space , $(\lambda,\gamma)^*$-continuous function .

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1 INTRODUCTION

THE study of semi open sets and semi continuity in topological spaces was initiated by Levine [5]. Analogous to the concept of generalized closed sets introduced by Levine [6], Bhattacharya and Lahiri [3] introduced the concept of semi generalized closed sets in topological spaces. Kasahara [4], defined the concept of an operation on topological spaces and introduced the concept of closed graphs of a function. Ahmad and Hussain [2], continued studying the properties of operations on topological spaces.

In this paper, we introduce new classes of sets called λ-open and generalized λ-closed sets in topological spaces and study some of their properties. By using these sets we define λ-$T_{1/2}$ space and introduce the concept of $(\lambda,\gamma)^*$-continuous functions and study some of their basic properties.

2 PRELIMINARIES

Throughout, X denote topological spaces. Let A be a subset of X , then the closure and the interior of A are denoted by $Cl(A)$ and $Int(A)$ respectively. A subset A of a topological space (X,τ) is said to be semi open [5] if $A\subseteq Cl(Int(A))$. The complement of a semi open set is said to be semi closed [5]. The family of all semi open (resp. semi closed) sets in a topological space (X,τ) is denoted by $SO(X,\tau)$ or $SO(X)$ (resp. $SC(X,\tau)$ or $SC(X)$). We consider λ as a function defined on $SO(X)$ into $P(X)$ and $\lambda:SO(X)\to P(X)$ is called an s-operation if $V\subseteq\lambda(V)$ for each non-empty semi open set V . It is assumed that $\lambda(\phi)=\phi$ and $\lambda(X)=X$ for any s-operation λ.

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3 λ- OPEN SET

Definition 3.1. [1] Let (X,τ) be a topological space and $\lambda:SO(X)\to P(X)$ be an s-operation, then a subset A of X is called a λ-open set if for each $x\in A$ there exists a semi open set U such that $x\in U$ and $\lambda(U)\subseteq A$.

The complement of a λ-open set is said to be λ-closed. The family of all λ-open (resp., λ-closed) subsets of a topological space (X,τ) is denoted by $SO_\lambda(X,\tau)$ or $SO_\lambda(X)$ (resp. , $SC_\lambda(X,\tau)$ or $SC_\lambda(X)$).

Proposition 3.2. For a topological space (X,τ), $SO_\lambda(X)\subseteq SO(X)$.

Proof. Obvious

The following examples show that the converse of the above proposition may not be true in general.

Example 3.3. Let $X=\{a,b,c\}$, and $\tau=\{\phi,\{a\},X\}$. We define an s-operation $\lambda:SO(X)\to P(X)$ as $\lambda(A)=A$ if $b\in A$ and $\lambda(A)=X$ otherwise. Here, we have $\{a,c\}$ is semi open set but it is not λ-open.

Definition 3.4. Let (X,τ) be a space, an s-operation λ is said to be s-regular if for every semi open sets U and V of $x\in X$, there exists a semi open set W containing x such that $\lambda(W)\subseteq\lambda(U)\cap\lambda(V)$.

Definition 3.5. Let (X,τ) be a topological space and let A be a subset of X. Then:

(1) The λ-closure of A ($\lambda Cl(A)$) is the intersection of all λ-closed sets containing A

(2) The λ-interior of A ($\lambda Int(A)$) is the union of all λ-open sets of X contained in A

(3) A point $x\in X$, is said to be a λ-limit point of A if every λ-open set containing X contains a point of A different from X, and the set of all λ-limit points of A is called the λ-derived set of A denoted by $\lambda d(A)$.

Proposition 3.6. For each point $x\in X$, $x\in\lambda Cl(A)$ if and only if $V\cap A\neq\phi$, for every $V\in SO_\lambda(X)$ such that $x\in V$.

Proof. Straightforward.

Proposition 3.7. Let $\{A_\alpha\}_{\alpha \in I}$ be any collection of λ-open sets in a topological space (X,τ), then $\bigcup A_\alpha$ is a λ-open set.
Proof. Let $x \in \bigcup_{\alpha \in I} A_\alpha$ then there exist $\alpha_0 \in I$ such that $x \in A_{\alpha 0}$, since A_α is a λ-open set for all $\alpha \in I$ implies that there exists a semi open set U such that $\lambda(U) \subseteq A_{\alpha_0} \subseteq \bigcup_{\alpha \in I} A_\alpha$. Therefore $\bigcup_{\alpha \in I} A_\alpha$ is a λ-open subset of (X,τ).

The following example shows that the intersection of two λ-open sets need not be λ-open.
Example 3.8. Let $X = \{a,b,c\}$ and $\tau = P(X)$. We define an s-operation $\lambda : SO(X) \to P(X)$ as $\lambda(A) = A$ if $A \neq \{a\},\{b\}$ and $\lambda(A) = X$ otherwise. Now, we have $\{a,b\}$ and $\{b,c\}$ are λ-open sets but $\{a,b\} \cap \{b,c\} = \{b\}$ is not λ-open.

Proposition 3.9. Let λ be an s-regular s-operation. If A and B are λ-open sets in X, then $A \cap B$ is also a λ-open set.
Proof. Let $x \in A \cap B$ then $x \in A$ and $x \in B$. Since A and B are λ-open sets, there exists semi open sets U and V such that $x \in U$ and $\lambda(U) \subseteq A$, $x \in V$ and $\lambda(V) \subseteq B$. Since λ is a s-regular s-operation, this implies there exists a semi open set W of X such that $\lambda(W) \subseteq \lambda(U) \cap \lambda(V) \subseteq A \cap B$. This implies that $A \cap B$ is λ-open .
Proposition 3.10. Let (X,τ) be a topological space and $A \subseteq X$. Then A is a λ-closed subset of X if and only if $\lambda d(A) \subseteq A$.
Proof. Obvious.
Proposition 3.11. For subsets A, B of a topological space (X,τ), the following statements are true.

(1) $A \subseteq \lambda Cl(A)$.

(2) $\lambda Cl(A)$ is λ-closed set in X.

(3) $\lambda Cl(A)$ is smallest λ-closed set which contain A.

(4) A is λ-closed set if and only if $A = \lambda Cl(A)$.

(5) $\lambda Cl(\phi) = \phi$ and $\lambda Cl(X) = X$.

(6) If A and B are subsets of space X with $A \subseteq B$. Then $\lambda Cl(A) \subseteq \lambda Cl(B)$.

(7) For any subsets A, B of a space (X,τ), $\lambda Cl(A) \cup \lambda Cl(B) \subseteq \lambda Cl(A \cup B)$.

(8) For any subsets A, B of a space (X,τ), $\lambda Cl(A \cap B) \subseteq \lambda Cl(A) \cap \lambda Cl(B)$.

Proof. Obvious.
Proposition 3.12. Let (X,τ) be a topological space and $A \subseteq X$. Then $\lambda Cl(A) = A \cup \lambda d(A)$.
Proof. Obvious.
Proposition 3.13. For a subset A of a topological space (X,τ), $\lambda Int(A) = A \setminus \lambda d(X \setminus A)$.
Proof. Obvious.
Proposition 3.13. For any subset A of a topological space X. The following statements are true.

(1) $X \setminus \lambda Int(A) = \lambda Cl(X \setminus A)$.

(2) $\lambda Cl(A) = X \setminus \lambda Int(X \setminus A)$.

(3) $X \setminus \lambda Cl(A) = \lambda Int(X \setminus A)$.

(4) $\lambda Int(A) = X \setminus \lambda Cl(X \setminus A)$.

Proof. Obvious.

Theorem 3.14. Let A, B be subsets of X. If $\lambda : SO(X) \to P(X)$ is an s-regular s-operation, then:

(1) $\lambda Cl(A \cup B) = \lambda Cl(A) \cup \lambda Cl(B)$.

(2) $\lambda Int(A \cap B) = \lambda Int(A) \cap \lambda Int(B)$.

Proof. Obvious.

4 Generalized λ-Closed Set and λ-$T_{1/2}$ Space

In this section, we define a new class of sets called generalized λ-closed set and we give some of its properties.
Definition 4.1. A subset A of a topological space (X,τ) is said to be generalized λ-closed (briefly. g-λ-closed) if $\lambda Cl(A) \subseteq U$, whenever $A \subseteq U$ and U is a λ-open set in (X,τ).

We say that a subset B of X is generalized λ-open (briefly. g-λ-open) if its complement $X \setminus B$ is generalized λ-closed in (X,τ).

In the following proposition we show every λ-closed subset of X is g-λ-closed.
Proposition 4.2. Every λ-closed set is g-λ-closed.
Proof. A set $A \subseteq X$ is λ-closed if and only if $\lambda Cl(A) = A$. Thus $\lambda Cl(A) \subseteq U$ for every $U \in SO_\lambda(X)$ containing A.

The reverse claim in the above proposition is not true in general. Next we give an example of a g-λ-closed set which is not λ-closed.

Example 4.3. Let $X = \{a,b,c\}$, and $\tau = P(X)$. We define an s-operation $\lambda : SO(X) \to P(X)$ as $\lambda(A) = A$ if $A = \{a\}$ and $\lambda(A) = X$ otherwise. Then, if we let $A = \{a,b\}$, and since the only λ-open supersets of A is X, so A is g-λ-closed but it is not λ-closed.

Proposition 4.4. The intersection of a g-λ-closed set and a λ-closed set is always g-λ-closed.
Proof. Let A be g-λ-closed and F be λ-closed. Assume that U is λ-open set such that $A \cap F \subseteq U$, set $G = X \setminus F$. Then $A \subseteq U \cup G$, since G is λ-open, then $U \cup G$ is λ-open and since A is g-λ-closed, then $\lambda Cl(A) \subseteq U \cup G$. Now by Proposition 3.8, $\lambda Cl(A \cap F) \subseteq \lambda Cl(A) \cap \lambda Cl(F) =$ $\lambda Cl(A) \cap F \subseteq (U \cup G) \cap F = (U \cap F) \cup (G \cap F) =$ $(U \cap F) \cup \phi \subseteq U$.

The union of two g-λ-closed sets need not be g-λ-closed, as it is shown in the following example:

Example 4.5. Let $X = \{a,b,c\}$, and $\tau = P(X)$. We define an s-operation $\lambda : SO(X) \to P(X)$ as:

$\lambda(A) = A$, *if* $A = \phi$ *or* $\{a,b\}$ *or* $\{a,c\}$ *or* $\{b,c\}$

$\lambda(A) = X$, otherwise

Then, if $A = \{a\}$ and $B = \{b\}$. So, A and B are g-λ-closed, but $A \cup B = \{a,b\}$ is not a g-λ-closed, since $\{a,b\}$ is λ-open and $\lambda Cl(\{a,b\}) = X$.
Theorem 4.6. If $\lambda : SO(X) \to P(X)$ is a s-regular s-operation. Then the finite union of g-λ-closed sets is always a g-λ-closed set.
Proof. Let A and B be two g-λ-closed sets, and let

$A \cup B \subseteq U$, where U is λ-open. Since A and B are g-λ-closed sets, therefore $\lambda Cl(A) \subseteq U$ and $\lambda Cl(B) \subseteq U$ implies that $\lambda Cl(A) \cup \lambda Cl(B) \subseteq U$. But by Theorem 3.11, we have $\lambda Cl(A) \cup \lambda Cl(B) = \lambda Cl(A \cup B)$. Therefore $\lambda Cl(A \cup B) \subseteq U$. Hence we get $A \cup B$ is g-λ-closed set.

The intersection of two g-λ-closed sets need not be g-λ-closed, as it is shown in the following example:

Example 4.7. Let $X = \{a,b,c\}$, and $\tau = P(X)$. We define an s-operation $\lambda : SO(X) \to P(X)$ as: $\lambda(A) = A$, *if* $A = \{a\}$ and $\lambda(A) = X$, *otherwise* .
Then the sets $A = \{a,b\}$ and $B = \{a,c\}$ are g-λ-closed sets, since X is their only λ-open superset. But $C = \{a\} = A \cap B$ is not g-λ-closed, since $C \subseteq \{a\} \in SO_\lambda(X)$ and $\lambda Cl(C) = X \not\subset \{a\}$.

Theorem 4.8. If a subset A of X is g-λ-closed and $A \subseteq B \subseteq \lambda Cl(A)$, then B is a g-λ-closed set in X .
Proof. Let A be g-λ-closed set such that $A \subseteq B \subseteq \lambda Cl(A)$. Let U be a λ-open set of X such that $B \subseteq U$. Since A is g-λ-closed, we have $\lambda Cl(A) \subseteq U$. Now $\lambda Cl(A) \subseteq \lambda Cl(B) \subseteq \lambda Cl(\lambda Cl(A)) = \lambda Cl(A) \subseteq U$. That is $\lambda Cl(B) \subseteq U$, where U is λ-open. Therefore B is a g-λ-closed set in X .

The converse of the Theorem 4.8 need not be true as seen from the following example.

Example 4.9. Let $X = \{a,b,c\}$, with $\tau = \{\phi, \{a\}, \{c\}, \{a,c\}, \{b,c\}, X\}$. Let $\lambda : SO(X) \to P(X)$ be a λ-identity s-operation. If $A = \{a\}$ and $B = \{a,b\}$. Then A and B are g-λ-closed sets in (X,τ). But $A \subseteq B$ Ù $\lambda Cl(A)$.

Theorem 4.10. Let $\lambda : SO(X) \to P(X)$ be an s-operation. Then for each $x \in X$, $\{x\}$ is λ-closed or $X \setminus \{x\}$ is g-λ-closed in (X,τ).
Proof. Suppose that $\{x\}$ is not λ-closed, then $X \setminus \{x\}$ is not λ-open. Let U be any λ-open set such that $X \setminus \{x\} \subseteq U$, then $U = X$. Therefore $\lambda Cl(X \setminus \{x\}) \subseteq U$. Hence $X \setminus \{x\}$ is g-λ-closed.

Proposition 4.11. A subset A of (X,τ) is g-λ-closed if and only if $\lambda Cl(\{x\}) \cap A \neq \phi$, holds for every $x \in \lambda Cl(A)$.
Proof. Let U be a λ-open set such that $A \subseteq U$ and let $x \in \lambda Cl(A)$. By assumption, there exists a point $z \in \lambda Cl(\{x\})$ and $z \in A \subseteq U$. It follows from Proposition 3.6, that $U \cap \{x\} \neq \phi$, hence $x \in U$, implies $\lambda Cl(A) \subseteq U$. Therefore A is g-λ-closed.
Conversely, suppose that $x \in \lambda Cl(A)$ such that $\lambda Cl(\{x\}) \cap A = \phi$. Since, $\lambda Cl(\{x\})$ is λ-closed. Therefore, $X \setminus \lambda Cl(\{x\})$ is λ-open set in X . Since $A \subseteq X \setminus \lambda Cl(\{x\})$ and A is g-λ-closed implies that $\lambda Cl(A) \subseteq X \setminus \lambda Cl(\{x\})$ holds and hence $x \notin \lambda Cl(A)$ a contradiction. Therefore $\lambda Cl(\{x\}) \cap A \neq \phi$.

Theorem 4.12. If a subset A of X is g-λ-closed set in X . Then $\lambda Cl(A) \setminus A$ does not contain any non empty λ-closed set in X .
Proof. Let A be a g-λ-closed set in X . We prove the result by contradiction. Let F be a λ-closed set such that $F \subseteq \lambda Cl(A) \setminus A$ and $F \neq \phi$. Then $F \subseteq X \setminus A$ which implies $A \subseteq X \setminus F$. Since A is g-λ-closed and $X \setminus F$ is λ-open set, therefore $\lambda Cl(A) \subseteq X \setminus F$, that is $F \subseteq X \setminus \lambda Cl(A)$.
Hence $F \subseteq \lambda Cl(A) \cap X \setminus \lambda Cl(A) = \phi$. This shows that $F = \phi$ which is a contradiction. Hence $\lambda Cl(A) \setminus A$ does not contains any non empty λ-closed set in X .

Lemma 4.13. Let A be a subset of a topological space (X,τ). If $\lambda d(A) \subseteq U$ for U is λ-open, then $\lambda d(\lambda d(A)) \subseteq U$, where λ is s-regular.
Proof. Suppose $x \in \lambda d(\lambda d(A))$ but $x \notin U$. Then $x \notin \lambda d(A)$ and so, for some λ-open set V, $x \in V$ and $A \cap V \subseteq \{x\}$, but $x \in \lambda d(\lambda d(A))$ implies that there exists $y \in \lambda d(A) \cap V \setminus \{x\}$. Now, $y \in U \cap V$ and $y \in \lambda d(A)$ and so $\phi \neq A \cap U \cap V \cap X \setminus \{y\} \subseteq A \cap V \subseteq \{x\}$. It follows that $x \in U$ which is contradiction.

Theorem 4.14. If λ is s-regular s-operation, then the λ-derived set is g-λ-closed.
Proof. If A is any subset of a topological space (X,τ) with $\lambda d(A) \subseteq U$ for U is λ-open. Then by Lemma 4.13 $\lambda Cl(\lambda d(A)) = \lambda d(\lambda d(A)) \cup \lambda d(A) \subseteq U$.

Theorem 4.15. A subset A of a topological space (X,τ) is g-λ-open if and only if $F \subseteq \lambda Int(A)$ whenever $F \subseteq A$ and F is λ-closed in (X,τ).
Proof. Let A be g-λ-open and $F \subseteq A$ where F is λ-closed. Since $X \setminus A$ is g-λ-closed and $X \setminus F$ is a λ-open set containing $X \setminus A$ implies $\lambda Cl(X \setminus A) \subseteq X \setminus F$. By Proposition 3.10, $X \setminus \lambda Int(A) \subseteq X \setminus F$. That is $F \subseteq \lambda Int(A)$.
Conversely, suppose that F is λ-closed and $F \subseteq A$, implies that $F \subseteq \lambda Int(A)$. Let $X \setminus A \subseteq U$, where U is λ-open. Then $X \setminus U \subseteq A$, where $X \setminus U$ is λ-closed. By hypothesis $X \setminus U \subseteq \lambda Int(A)$. That is $X \setminus \lambda Int(A) \subseteq U$ and then by Proposition 3.10, $\lambda Cl(X \setminus A) \subseteq U$. This implies $X \setminus A$ is g-λ-closed and A is g-λ-open.

The union of two g-λ-open sets need not be g-λ-open. As it is shown in the following example:

Example 4.16. Let $X = \{a,b,c\}$, and $\tau = P(X)$. We define an s-operation $\lambda : SO(X) \to P(X)$ as $\lambda(A) = A$ if $A = \{b\}$ and $\lambda(A) = X$ if $A \neq \{b\}$. If $A = \{a\}$ and $B = \{c\}$, then A and B are g-λ-open sets in X, but $A \cup B = \{a,c\}$ $B = \{a,c\}$ is not a g-λ-open set in X .

Theorem 4.17. Let $\lambda : SO(X) \to P(X)$ be a s-regular s-operation and let A and B be two g-λ-open sets in a space X , then $A \cap B$ is also g-λ-open.
Proof. If A and B are g-λ-open sets in a space X . Then $X \setminus A$ and $X \setminus B$ are g-λ-closed sets in a space X . By Theorem 4.6, $X \setminus A \cup X \setminus B$ is also g-λ-closed set in X . That is $X \setminus A \cup X \setminus B = X \setminus (A \cap B)$ is a g-λ-closed set in X . Therefore $A \cap B$ is a g-λ-open set in X .

Theorem 4.18. A set A is g-λ-open if and only if $\lambda Int(A) \cup X \setminus A \subseteq G$ and G is λ-open implies $G = X$.
Proof. Suppose that A is g-λ-open in X . Let G be λ-open and $\lambda Int(A) \cup X \setminus A \subseteq G$. This implies

$X \setminus G \subseteq X \setminus (\lambda Int(A) \cup X \setminus A) = X \setminus \lambda Int(A) \cap A$. That is $X \setminus G \subseteq (X \setminus \lambda Int(A)) \setminus (X \setminus A)$. Thus $X \setminus G \subseteq \lambda Cl(X \setminus A) \setminus (X \setminus A)$, since $X \setminus \lambda Int(A) = \lambda Cl(X \setminus A)$. Now, $X \setminus G$ is λ-closed and $X \setminus A$ is g-λ-closed, by Theorem 4.12, it follows that $X \setminus G = \phi$. Hence $G = X$.

Conversely, let $\lambda Int(A) \cup X \setminus A \subseteq G$ and G is λ-open, this implies that $G = X$. Let U be a λ-open set such that $X \setminus A \subseteq U$. Now $\lambda Int(A) \cup X \setminus A \subseteq \lambda Int(A) \cup U$ which is clearly, λ-open and so by the given condition $\lambda Int(A) \cup U = X$, which implies that $\lambda Cl(X \setminus A) \subseteq U$. Hence $X \setminus A$ is g-λ-closed, therefore A is g-λ-open.

Theorem 4.19. Every singleton set in a space X is either g-λ-open or λ-closed.

Proof: Suppose that $\{x\}$ is not g-λ-open, then by definition $X \setminus \{x\}$ is not g-λ-closed. This implies that by Theorem 4.10, the set $\{x\}$ is λ-closed.

Theorem 4.20. If $\lambda Int(A) \subseteq B \subseteq A$ and A is g-λ-open, then B is g-λ-open.

Proof. $\lambda Int(A) \subseteq B \subseteq A$ implies $X \setminus A \subseteq X \setminus B \subseteq X \setminus \lambda Int(A)$. That is, $X \setminus A \subseteq X \setminus B \subseteq \lambda Cl(X \setminus A)$ by Proposition 3.10. Since $X \setminus A$ is g-λ-closed, by Theorem 4.8, $X \setminus B$ is g-λ-closed and B is λ-open.

Theorem 4.21. Let (X, τ) be a topological space (X, τ) and $\lambda : SO(X) \to P(X)$ be an s-operation. The space (X, τ) is λ-$T_{1/2}$ if and only if Each singleton $\{x\}$ of X is either λ-closed set or λ-open set.

Proof. Suppose $\{x\}$ is not λ-closed. Then by Proposition 4.10, $X \setminus \{x\}$ is g-λ-closed. Now since (X, τ) is λ-$T_{1/2}$, $X \setminus \{x\}$ is λ-closed i.e. $\{x\}$ is λ-open.

Conversely. Let A be any g-λ-closed set in (X, τ) and $x \in \lambda Cl(A)$. By (2) we have $\{x\}$ is λ-closed or λ-open. If $\{x\}$ is λ-closed then $x \notin A$ will imply $x \in \lambda Cl(A) \setminus A$ which is not possible by Proposition 4.12. Hence $x \in A$. Therefore, $\lambda Cl(A) = A$, i.e. A is λ-closed. So (X, τ) is λ-$T_{1/2}$. On the other hand, if $\{x\}$ is λ-open then as $x \in \lambda Cl(A)$, $\{x\} \cap A \neq \phi$. Hence $x \notin A$. So A is λ-closed.

5 $(\lambda,\gamma)^*$-Continuous and $(\lambda,\gamma)^*$-Open Functions

In this section, some types of continuous functions via s-operations are introduced and investigated. Several properties of these functions are obtained.

Throughout, (X, τ), (Z, ρ) and (Y, σ) are topological spaces and λ, η and γ are s-operations on the family of semi open sets of the topological spaces respectively.

Definition 5.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $(\lambda,\gamma)^*$-continuous, if for each x of X and each γ-open set V of Y containing $f(x)$, there exists a λ-open set U of X such that $x \in U$ and $f(U) \subseteq V$.

Theorem 5.2. Let $f : (X, \tau) \to (Y, \sigma)$ be a function, then f is $(\lambda,\gamma)^*$-continuous if and only if for each γ-open set B in Y, $f^{-1}(B)$ is λ-open in X.

Proof. Let f be a $(\lambda,\gamma)^*$-continuous and $B \in SO_\gamma(Y)$, let $A = f^{-1}(B)$. We show that A is λ-open in X. For this, let $x \in A$, then it implies that $f(x) \in B$. Hence, by hypothesis, there exists $A_x \in SO_\lambda(X)$ such that $x \in A_x$ and $f(A_x) \subseteq B$. Then $A_x \subseteq f^{-1}(f(A_x)) \subseteq f^{-1}(B) = A$. Thus $A = \bigcup \{A_x : x \in A\}$. It follows that A is λ-open in X.

Conversely, let $x \in X$ and $B \in SO_\gamma(Y)$ such that $f(x) \in B$. Let $A = f^{-1}(B)$. By hypothesis, A is λ-open in X and also we have $x \in f^{-1}(B) = A$ as $f(x) \in B$. Thus, $f(A) = f(f^{-1}(B)) \subseteq B$. Hence f is $(\lambda,\gamma)^*$-continuous.

Theorem 5.3. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the following statements are equivalent:

(1) f is $(\lambda,\gamma)^*$-continuous.

(2) The inverse image of each γ-closed set in Y is a λ-closed set in X.

(3) $\lambda Cl(f^{-1}(V)) \subseteq f^{-1}(\gamma Cl(V))$, for every $V \subseteq Y$.

(4) $f(\lambda Cl(U)) \subseteq \gamma Cl(f(U))$, for every $U \subseteq X$.

(5) $\lambda Bd(f^{-1}(V)) \subseteq f^{-1}(\gamma Bd(V))$, for every $V \subseteq Y$.

(6) $f(\lambda d(U)) \subseteq \gamma Cl(f(U))$, for every $U \subseteq X$.

(7) $f^{-1}(\gamma Int(V)) \subseteq \lambda Int(f^{-1}(V))$, for every $V \subseteq Y$.

Proof. **(1)$\Longrightarrow$(2):** Let $F \subseteq Y$ be γ-closed. Since f is $(\lambda,\gamma)^*$-continuous, $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is λ-open. Therefore, $f^{-1}(F)$ is λ-closed in X.

(2)$\Longrightarrow$(3): Since $\gamma Cl(V)$ is γ-closed for every $V \subseteq Y$, then $f^{-1}(\gamma Cl(V))$ is λ-closed. Therefore $\lambda Cl(f^{-1}(V)) \subseteq \lambda Cl(f^{-1}(\gamma Cl(V))) = f^{-1}(\gamma Cl(V))$.

(3)$\Longrightarrow$(4): Let $U \subseteq X$ and $f(U) = V$. Then $\lambda Cl(f^{-1}(V)) \subseteq f^{-1}(\gamma Cl(V))$. Thus $\lambda Cl(U) \subseteq \lambda Cl(f^{-1}(f(U))) \subseteq f^{-1}(\gamma Cl(f(U)))$ then we get $f(\lambda Cl(U)) \subseteq \gamma Cl(f(U))$.

(4)$\Longrightarrow$(2): Let $W \subseteq Y$ be a γ-closed set and $U = f^{-1}(W)$. This implies that $f(\lambda Cl(U)) \subseteq \gamma Cl(f(U)) = \gamma Cl(f(f^{-1}(W))) \subseteq \gamma Cl(W) = W$.

Thus $\lambda Cl(U) \subseteq f^{-1}(f(\lambda Cl(U))) \subseteq f^{-1}(W) = U$. So U is λ-closed.

(2)$\Longrightarrow$(1): Let $V \subseteq Y$ be an γ-open set, then $Y \setminus V$ is γ-closed. Hence, $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is λ-closed in X and so $f^{-1}(V)$ is γ-open in X.

(5)$\Longrightarrow$(7): Let $V \subseteq Y$, then by hypothesis, $\lambda Bd(f^{-1}(V)) \subseteq f^{-1}(\gamma Bd(V))$. This implies that $f^{-1}(V) \setminus \lambda Int(f^{-1}(V)) \subseteq f^{-1}(V \setminus \gamma Int(V)) = f^{-1}(V) \setminus f^{-1}(\gamma Int(V))$. Hence we get $f^{-1}(\gamma Int(V)) \subseteq \lambda Int(f^{-1}(V))$.

(7)$\Longrightarrow$(5): Let $V \subseteq Y$, then by hypothesis, $f^{-1}(\gamma Int(V)) \subseteq \lambda Int(f^{-1}(V))$. Implies that $f^{-1}(V) \setminus \lambda Int(f^{-1}(V)) \subseteq f^{-1}(V) \setminus f^{-1}(\gamma Int(V))$ then $\lambda Bd(f^{-1}(V)) \subseteq f^{-1}(\gamma Bd(V))$.

(1)$\Longrightarrow$(6): Since f is $(\lambda,\gamma)^*$-continuous and by (4), we have $f(\lambda Cl(U)) \subseteq \gamma Cl(f(U))$ for each $U \subseteq X$. So $f(\lambda Cl(U)) \subseteq \gamma Cl(f(U))$.

(6) $\Longrightarrow$ (1): Let V be a γ-closed subset of Y and let $f^{-1}(V) = W$, then by hypothesis, $f(\lambda Cl(W)) \subseteq \gamma Cl(f(W))$. Thus $f(\lambda Cl(f^{-1}(V))) \subseteq \gamma Cl(f(f^{-1}(V))) \subseteq \gamma Cl(V) = V$. Hence, $\lambda Cl(f^{-1}(V)) \subseteq f^{-1}(V)$ so by Proposition 3.4, $f^{-1}(V)$ is λ-closed set. Therefore, by part (2) of this theorem f is $(\lambda,\gamma)^*$-continuous.

(1) $\Longrightarrow$ (7): Let $V \subseteq Y$, then $f^{-1}(\gamma Int(V))$ is λ-open set in X. Thus $f^{-1}(\gamma Int(V)) = \lambda Int\, f^{-1}(\gamma Int(V)) \subseteq \lambda Int(f^{-1}(V))$. Therefore, $f^{-1}(\gamma Int(V)) \subseteq \lambda Int(f^{-1}(V))$.

(7) $\Longrightarrow$ (1): Let $V \subseteq Y$ be an γ-open set. Then $f^{-1}(V) = f^{-1}(\gamma Int(V)) \subseteq \lambda Int(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is λ-open. Hence f is $(\lambda,\gamma)^*$-continuous.

Proposition 5.4. If the functions $f:(X,\tau) \to (Z,\rho)$ is $(\lambda,\eta)^*$-continuous and $g:(Z,\rho) \to (Y,\sigma)$ is $(\eta,\gamma)^*$-continuous, then $g \circ f:(X,\tau) \to (Y,\sigma)$ is $(\lambda,\gamma)^*$-continuous.

Proof. Let $V \in SO_\gamma(Y)$. Then $g^{-1}(V) \in SO_\eta(Z)$ and $f^{-1}(g^{-1}(V)) \in SO_\lambda(X)$. This implies that $(g \circ f)^{-1}(V) \in SO_\lambda(X)$. Therefore, $g \circ f:(X,\tau) \to (Y,\sigma)$ is $(\lambda,\gamma)^*$-continuous.

Definition 5.5. A function $f:(X,\tau) \to (Y,\sigma)$ is said to be $(\lambda,\gamma)^*$-open ($(\lambda,\gamma)^*$-closed), if for any λ-open (λ-closed) set A of (X,τ), $f(A)$ is γ-open (γ-closed).

Theorem 5.6. Suppose that $f:(X,\tau) \to (Y,\sigma)$ is $(\lambda,\gamma)^*$-continuous and $(\lambda,\gamma)^*$-closed function, then:

(1) For every g-λ-closed set A of (X,τ) the image $f(A)$ is a g-γ-closed set.

(2) For every g-γ-closed set B of (Y,σ) the inverse set $f^{-1}(B)$ is a g-λ-closed set.

Proof. (1) Let V be any γ-open set in (Y,σ) such that $f(A) \subseteq V$. Then by Theorem 5.2, $f^{-1}(V)$ is λ-open. Since A is g-λ-closed and $A \subseteq f^{-1}(V)$, we have $\lambda Cl(A) \subseteq f^{-1}(V)$ and hence we get $f(\lambda Cl(A)) \subseteq V$. By assumption $f(\lambda Cl(A))$ is a γ-closed set. Therefore, $\gamma Cl(f(A)) \subseteq \gamma Cl(f(\lambda Cl(A))) = f(\lambda Cl(A)) \subseteq V$. This implies that $f(A)$ is g-γ-closed.

(2) Let U be any λ-open set such that $f^{-1}(B) \subseteq U$. Let $H = \lambda Cl(f^{-1}(B)) \cap (X \setminus U)$. Then H is λ-closed in (X,τ). This implies $f(H)$ is γ-closed set in Y. Since $f(H) = f(\lambda Cl(f^{-1}(B)) \cap X \setminus U) \subseteq \gamma Cl(B) \cap f(X \setminus U) \subseteq \gamma Cl(B) \cap (Y \setminus B)$. This implies that $f(H) = \phi$ and since f is a function, hence $H = \phi$. Therefore, $\lambda Cl(f^{-1}(B)) \subseteq U$. This implies $f^{-1}(B)$ is g-λ-closed.

Theorem 5.7. A function $f:(X,\tau) \to (Y,\sigma)$ is $(\lambda,\gamma)^*$-open if and only if $f(\lambda Int(A)) \subseteq \gamma Int(f(A))$ for all $A \subseteq X$.

Proof. Let $A \subseteq X$ and let $x \in \lambda Int(A)$. Then there exists $U_x \in SO_\lambda(X)$ such that $x \in U_x \subseteq A$. So $f(x) \in f(U_x) \subseteq f(A)$ and by hypothesis, $f(U_x) \in SO_\gamma(Y)$. Hence $f(x) \in \gamma Int(f(A))$. Thus $f(\lambda Int(A)) \subseteq \gamma Int(f(A))$.

Conversely, let $U \in SO_\lambda(X)$. Then by hypothesis, we get $f(\lambda Int(U)) \subseteq \gamma Int(f(U))$. Since $\lambda Int(U) = U$ as U is λ-open. Also $\gamma Int(f(U)) \subseteq f(U)$. Hence $f(U) = \gamma Int(f(U))$. Thus $f(U)$ is γ-open in Y. So f is $(\lambda,\gamma)^*$-open.

Theorem 5.8. A function $f:(X,\tau) \to (Y,\sigma)$ is $(\lambda,\gamma)^*$-open if and only if $\lambda Int(f^{-1}(B)) \subseteq f^{-1}(\gamma Int(B))$ for all $B \subseteq Y$.

Proof. Let $B \subseteq Y$, since $\lambda Int(f^{-1}(B))$ is λ-open set in X and f is $(\lambda,\gamma)^*$-open function, so $f(\lambda Int(f^{-1}(B)))$ is γ-open set in Y. We have $f(\lambda Int(f^{-1}(B))) \subseteq f(f^{-1}(B)) \subseteq B$. Hence $f(\lambda Int(f^{-1}(B))) \subseteq \gamma Int(B)$ by hypothesis. Therefore $\lambda Int(f^{-1}(B)) \subseteq f^{-1}(\gamma Int(B))$.

Conversely, let $A \subseteq X$, then $f(A) \subseteq Y$. Hence by hypothesis, we obtain $\lambda Int(A) \subseteq \lambda Int(f^{-1}(f(A))) \subseteq f^{-1}(\gamma Int(f(A)))$. Implies that $f(\lambda Int(A)) \subseteq f(f^{-1}(\gamma Int(f(A)))) \subseteq \gamma Int(f(A))$. Thus $f(\lambda Int(A)) \subseteq \gamma Int(f(A))$, for all $A \subseteq X$. Hence, by Theorem 5.7, f is $(\lambda,\gamma)^*$-open.

Theorem 5.9. A function $f:(X,\tau) \to (Y,\sigma)$ is $(\lambda,\gamma)^*$-open if and only if $f^{-1}(\gamma Cl(B)) \subseteq \lambda Cl(f^{-1}(B))$ for every subset B of Y.

Proof. Let $B \subseteq Y$ and let $x \in f^{-1}(\gamma Cl(B))$, then $f(x) \in \gamma Cl(B)$. Let $U \in SO_\lambda(X)$ such that $x \in U$. By hypothesis, $f(U) \in SO_\gamma(Y)$ and $f(x) \in f(U)$. Thus $f(U) \cap B \neq \phi$ and hence $U \cap f^{-1}(B) \neq \phi$. Therefore, $x \in \lambda Cl(f^{-1}(B))$. So we obtain $f^{-1}(\gamma Cl(B)) \subseteq \lambda Cl(f^{-1}(B))$.

Conversely, let $B \subseteq Y$, then $(Y \setminus B) \subseteq Y$. By hypothesis, $f^{-1}(\gamma Cl(Y \setminus B)) \subseteq \lambda Cl(f^{-1}(Y \setminus B))$. Implies that $X \setminus \lambda Cl(f^{-1}(Y \setminus B)) \subseteq X \setminus f^{-1}(\gamma Cl(Y \setminus B))$. Hence $X \setminus \lambda Cl(X \setminus f^{-1}(B)) \subseteq X \setminus f^{-1}(Y \setminus \gamma Int(B))$. Then $\lambda Int(f^{-1}(B)) \subseteq f^{-1}(\gamma Int(B))$. Now by Theorem 5.8, it follows that f is $(\lambda,\gamma)^*$-open.

Theorem 5.10. Let $f:(X,\tau) \to (Z,\rho)$ and $g:(Z,\rho) \to (Y,\sigma)$ be two functions such that $g \circ f:(X,\tau) \to (Y,\sigma)$ is $(\lambda,\gamma)^*$-continuous. Then:

(1) If g is a $(\eta,\gamma)^*$-open injection, then f is $(\lambda,\eta)^*$-continuous.

(2) If f is a $(\lambda,\eta)^*$-open surjection, then g is $(\eta,\gamma)^*$-continuous.

Proof. (1) Let $U \in SO_\eta(Z)$. Since g is $(\eta,\gamma)^*$-open, then $g(U) \in SO_\gamma(Y)$. Also since $g \circ f$ is $(\lambda,\gamma)^*$-continuous. Therefore, we have $(g \circ f)^{-1}(g(U)) \in SO_\lambda(X)$. Since g is an injection function, so we have $(g \circ f)^{-1}(g(U)) = (f^{-1} \circ g^{-1})(g(U)) = (f^{-1})(g^{-1}(g(U))) = f^{-1}(U)$. Consequently $f^{-1}(U)$ is λ-open in X. This proves that f is $(\lambda,\eta)^*$-continuous.

(2) Let $V \in SO_\gamma(Y)$. Then $(g \circ f)^{-1}(V) \in SO_\lambda(X)$ since

$g \circ f$ is $(\lambda,\gamma)^*$-continuous. Also f is $(\lambda,\eta)^*$-open, $f((g\circ f)^{-1}(V))$ is η-open in Y. Since f is surjective, then:
$f((g\circ f)^{-1}(V)) = (f \circ (g\circ f)^{-1})(V) = (f \circ (f^{-1}\circ g^{-1}))(V) = ((f \circ f^{-1})\circ g^{-1})(V) = g^{-1}(V)$.
Hence g is $(\eta,\gamma)^*$-continuous.

Theorem 5.11. Let $f:(X,\tau)\to(Z,\rho)$ and $g:(Z,\rho)\to(Y,\sigma)$ are $(\lambda,\eta)^*$-closed (resp. open) and $(\eta,\gamma)^*$-closed (resp. open) respectively. Then the composition function $g\circ f:(X,\tau)\to(Y,\sigma)$ is a $(\lambda,\gamma)^*$-closed (resp., $(\lambda,\gamma)^*$-open) function.
Proof. Obvious.

Theorem 5.12. A function $f:(X,\tau)\to(Y,\sigma)$ is $(\lambda,\gamma)^*$-closed if and only if $\gamma Cl(f(A))\subseteq f(\lambda Cl(A))$, for every subset A of X.
Proof. Suppose f is a $(\lambda,\gamma)^*$-closed function and A is an arbitrary subset of X. Then $f(\lambda Cl(A))$ is γ-closed set in Y. Since $f(A)\subseteq f(\lambda Cl(A))$, we obtain $\gamma Cl(f(A))\subseteq f(\lambda Cl(A))$.
Conversely, suppose F is an arbitrary λ-closed set in X. By hypothesis, we obtain $f(F)\subseteq \gamma Cl(f(F))\subseteq f(\lambda Cl(F)) = f(F)$.
Hence $\gamma Cl(f(F)) = f(F)$. Thus $f(F)$ is γ-closed in Y. It follows that f is $(\lambda,\gamma)^*$-closed.

Theorem 5.13. Let $f:(X,\tau)\to(Y,\sigma)$ be a bijective function. Then the following statements are equivalent:

(1) f is $(\lambda,\gamma)^*$-closed.

(2) f is $(\lambda,\gamma)^*$-open.

(3) f^{-1} is $(\gamma,\lambda)^*$-continuous.

Proof. $(1)\Longrightarrow(2)$: Let $U\in SO_\lambda(X)$. Then $X\setminus U$ is λ-closed in X. By(1), $f(X\setminus U)$ is γ-closed in Y. But $f(X\setminus U) = f(X)\setminus f(U) = Y\setminus f(U)$. Thus $f(U)$ is γ-open in Y. This shows that f is $(\lambda,\gamma)^*$-open.
$(2)\Longrightarrow(3)$: Let A be a subset of X. Since f is $(\lambda,\gamma)^*$-open, so by Theorem 5.12, $f^{-1}(\gamma Cl(f(A)))\subseteq \lambda Cl(f^{-1}(f(A)))$. This implies that $\gamma Cl(f(A))\subseteq f(\lambda Cl(A))$. Thus $\gamma Cl((f^{-1})^{-1}(A))\subseteq (f^{-1})^{-1}(\lambda Cl(A))$, for all $A\subseteq X$. Then by Theorem 3.1.6, it follows that f^{-1} is $(\gamma,\lambda)^*$-continuous.
$(3)\Longrightarrow(1)$: Let A be an arbitrary λ-closed subset of X. Since f^{-1} is a $(\gamma,\lambda)^*$-continuous. Then by Theorem 3.1.6, $(f^{-1})^{-1}(A)$ is γ-closed in Y. But $(f^{-1})^{-1}(A) = f(A)$. This means that f is $(\gamma,\lambda)^*$-closed.

Definition 5.14. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be $(\lambda,\gamma)^*$-homeomorphism if it is bijective, $(\lambda,\gamma)^*$-continuous and $(\lambda,\gamma)^*$-open.
Corollary 5.15. If $f:(X,\tau)\to(Y,\sigma)$ is a bijective function, then the following statement are equivalent.

(1) f is $(\lambda,\gamma)^*$-homeomorphism.

(2) $f(\lambda Cl(A)) = \gamma Cl(f(A))$ for all $A\subseteq X$.

(3) $\lambda Cl(f^{-1}(B)) = f^{-1}(\gamma Cl(B))$ for all $B\subseteq Y$.

(4) $f(\lambda Int(A)) = \gamma Int(f(A))$ for all $A\subseteq X$.

(5) $\lambda Int(f^{-1}(B)) = f^{-1}(\gamma Int(B))$ for all $B\subseteq Y$.

Proof. $(1)\Longleftrightarrow(2)$. Obvious. Follows from Theorem 5.3 and Theorem 5.12.
$(1)\Longleftrightarrow(3)$. Follows from Theorem 5.3 and Theorem 5.9.
$(1)\Longleftrightarrow(5)$. Follows from Theorem 5.3 and Theorem 5.8.
$(1)\Longleftrightarrow(4)$. We have $\lambda Int(A) = X\setminus \lambda Cl(X\setminus A)$. Thus
$f(\lambda Int(A)) = Y\setminus \gamma Cl(f(X\setminus A)) = Y\setminus \gamma Cl(Y\setminus f(A)) = \gamma Int(f(A))$.

Theorem 5.16. Let $f:(X,\tau)\to(Y,\sigma)$ be a $(\lambda,\gamma)^*$-continuous and $(\lambda,\gamma)^*$-closed function. Then:

(1) If f is injective and (Y,σ) is a γ-$T_{1/2}$ space, then (X,τ) is a λ-$T_{1/2}$ space.

(2) If f is surjective and (X,τ) is a λ-$T_{1/2}$ space, then (Y,σ) is a γ-$T_{1/2}$ space.

Proof. (1) Let A be a g-λ-closed set in (X,τ). To show that A is λ-closed. By Theorem 5.6, we have $f(A)$ is g-γ-closed. Since (Y,σ) is γ-$T_{1/2}$, $f(A)$ is a γ-closed set. Since f is injective and $(\lambda,\gamma)^*$-continuous, $f^{-1}(f(A)) = A$ is a λ-closed set in X. Hence (X,τ) is a λ-$T_{1/2}$ space.
(2) Let B be a g-γ-closed set in (Y,σ). By Theorem 5.6, $f^{-1}(B)$ is g-λ- closed. Since (X,τ) is a λ-$T_{1/2}$ space, $f^{-1}(B)$ is λ-closed. Since f is surjective and $(\lambda,\gamma)^*$-continuous, $f(f^{-1}(B)) = B$ is a γ-closed set in Y. Therefore (Y,σ) is γ-$T_{1/2}$.

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